Review Handout For Math 2280

Trigonometric Identities	
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)]$	$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$
$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$	$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$	$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\cos x - \cos y = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y-x}{2}\right)$

Here, ω is the lower case Greek letter omega.

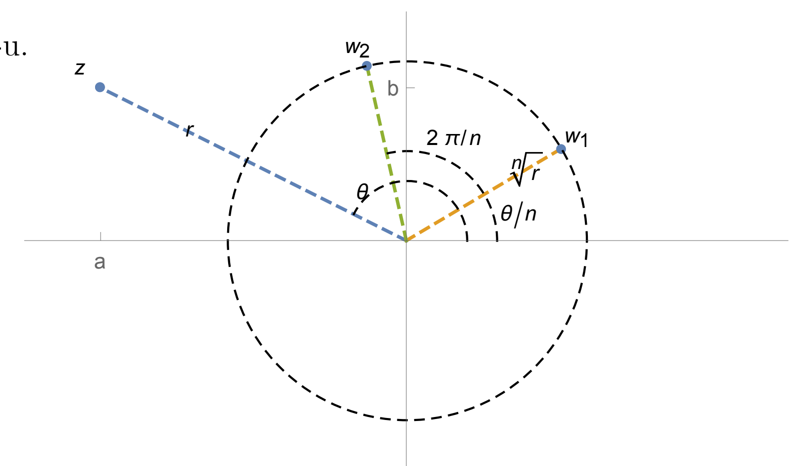
$A\cos(\omega t) + B\sin(\omega t) = C\sin(\omega t + \phi)$ where $C = \sqrt{A^2+B^2}$ and ϕ is the angle such that $\sin\phi = \frac{A}{C}$ and $\cos\phi = \frac{B}{C}$, or $\phi = \begin{cases} \tan^{-1}(\frac{A}{B}) & \text{, if } B > 0 \\ \pi + \tan^{-1}(\frac{A}{B}) & \text{, if } B < 0 \end{cases}$.

Complex Exponential (Euler's Formula):	$e^{i\theta} = \cos\theta + i\sin\theta$
Polar Form of Complex Numbers:	$z = a + b\,i = r\,e^{i\theta} = r\cos\theta + i\,r\sin\theta$ where $r = \sqrt{a^2+b^2}$ and θ is the angle such that $\sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$, or $\theta = \begin{cases} \tan^{-1}(\frac{b}{a}) & \text{, if } a > 0 \\ \pi + \tan^{-1}(\frac{b}{a}) & \text{, if } a < 0 \end{cases}$.
De Moivre's Formula:	For $z = a + b\,i = r\,e^{i\theta}$, its nth power is $z^n = r^n e^{in\theta} = r^n\cos(n\theta) + i\,r^n\sin(n\theta)$.

Here, w is the lower case of the letter double-u.

The n n-th roots of $z = a + b\,i = r\,e^{i\theta}$, are $w_k = \sqrt[n]{r}\,e^{i\left(\frac{\theta+2(k-1)\pi}{n}\right)} = \sqrt[n]{r}\cos\left(\frac{\theta+2(k-1)\pi}{n}\right) + i\sqrt[n]{r}\sin\left(\frac{\theta+2(k-1)\pi}{n}\right)$ for $k = 1, \cdots, n$.

Or, the solutions of $w^n = a + b\,i = r\,e^{i\theta}$ are $w_k = \sqrt[n]{r}\,e^{i\left(\frac{\theta+2(k-1)\pi}{n}\right)} = \sqrt[n]{r}\cos\left(\frac{\theta+2(k-1)\pi}{n}\right) + i\sqrt[n]{r}\sin\left(\frac{\theta+2(k-1)\pi}{n}\right)$ for $k = 1, \cdots, n$.



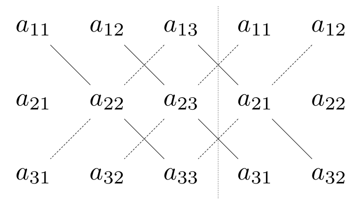
In words, the n n-th roots of $r\,e^{i\theta}$ are on a circle of radius $\sqrt[n]{r}$ and are $\frac{2\pi}{n}$ radians apart with the first one having the angle $\frac{\theta}{n}$.

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The determinant of the three columns on the left is the sum of the products along the solid diagonals minus the sum of the products along the dashed diagonals. *



For matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & \cdots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$, let A_{ij} be the matrix with the ith row and jth column removed:

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}.$$

Then, for any row i, $\text{Det}A = |A| = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\, |A_{ij}|$.

Also, for any column j, $\text{Det}A = |A| = \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij}\, |A_{ij}|$.

* The picture in the right is from Wikipedia.

Differentiation Rules	
$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = n\,x^{n-1}$
$\frac{d}{dx}[cf(x)] = cf'(x)$	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$
$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$	
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}a^x = (\ln a)\,a^x$
$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$	$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)\,x}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

Fundamental Theorem of Calculus: For a continuous function f, $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

Strategies For Integration

Method	Example
Expanding	$(e^x - e^{-x})^2 = e^{2x} - 2 + e^{-2x}$
Completing the square	$\sqrt{8x - x^2} = \sqrt{16 - (x-4)^2}$
Using a trigonometric identity	$\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$
Eliminating a square root	$\sqrt{x^2 + 2 + \frac{1}{x^2}} = \sqrt{\left(x + \frac{1}{x}\right)^2} = \left\lvert x + \frac{1}{x}\right\rvert$
Reducing an improper fraction	$\frac{x^3 - 7x}{x-2} = x^2 + 2x - 3 - \frac{6}{x-2}$
Separating a fraction	$\frac{3x+2}{1-x^2} = \frac{3x}{1-x^2} + \frac{2}{1-x^2}$
Multiplying by a form of 1	$\sec x = \sec x \times \frac{\sec x + \tan x}{\sec x + \tan x} = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

Integration Techniques

Method	Example
u-substitution $\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$	For $u = x^2+1$ we have $x\,dx = \frac{du}{2}$ and so $\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{du}{2\sqrt{u}}$
Integration by parts $\int u\,dv = uv - \int v\,du$	For $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ we have and so $\int x\,e^x\,dx = x\,e^x - \int e^x\,dx$
Trigonometric Substitution For expressions $\pm x^2 \pm a^2$	For $x = \tan\theta$ with $-\pi/2 < \theta < \pi/2$ we have $\sec\theta > 0$, $dx = \sec^2\theta\,d\theta$ and $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \lvert\sec\theta\rvert = \sec\theta$ Thus $\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta$
Partial Fractions $\int \frac{P(x)}{D(x)}\,dx$ where P and D are polynomials with deg $P <$ deg D. Factor $D(x)$.	$\int \frac{dx}{x^3+x^2} = \int \frac{dx}{x^2(x+1)} = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx$

Table of Integration Formulas

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, for $n \neq 1$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \frac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$	$\int \csc x\,dx = \ln\lvert\csc x - \cot x\rvert + C$
$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$	$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\lvert x + \sqrt{x^2\pm a^2}\rvert + C$
$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$	$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2+a^2}\right) + C$

For the power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ exactly one of the following three cases will hold and for each case a radius of convergence ρ is defined.

1. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges only for $x = x_0$ and $\rho = 0$.
2. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges for all x and $\rho = \infty$.
3. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converges if $|x-x_0| < R$ and diverges if $|x-x_0| > R$, for some positive number R, and $\rho = R$.

The radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ can be found as follows. Let $a_n = c_n(x-x_0)^n$ and consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

1. If $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| < 1$ holds only for $x = x_0$, then $\rho = 0$ and the interval of convergence is $\{x_0\}$.
2. If $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| < 1$ holds for all x, then $\rho = \infty$ and the interval of convergence is $(-\infty, \infty)$.
3. $\rho > 0$ is the radius of convergence if $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| < 1$ holds for $|x-x_0| < \rho$ and the interval of convergence is $(x_0-\rho, x_0+\rho)$ plus none, one or both endpoints $x = x_0 \pm \rho$ which must be checked individually.

Equivalently, the radius of convergence can be found as follows. If, $c_n \neq 0$ for large n, and $\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = L$, where $0 \leq L \leq \infty$, then $\rho = L$. In this version, for L positive and finite, the interval of convergence is $(x_0 - L, x_0 + L)$ plus none, one or both endpoints $x = x_0 \pm L$ which must be checked individually.